

Phase Transitions in a Probabilistic Cellular Automaton with Two Absorbing States

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Abstract

We study the phase diagram and the critical behavior of a one-dimensional radius-1 two-state totalistic probabilistic cellular automaton having two absorbing states. This system exhibits a first-order phase transition between the fully occupied state and the empty state, two second-order phase transitions between a partially occupied state and either the fully occupied state or the empty state, and a second-order damage-spreading phase transition. It is found that all the second-order phase transitions have the same critical behavior as the directed percolation model. The mean-field approximation gives a rather good qualitative description of all these phase transitions.

PACS: 05.45, 05.90, 03.20

1 Introduction

Probabilistic cellular automata (PCA) have been widely used to model a variety of systems with local interactions in physics, chemistry, biology and social sciences [1, 2, 3, 4, 5]. Many of these models exhibit transcritical bifurcations which may be viewed as second-order phase transitions. It has been conjectured [6, 7] that all second-order phase transitions from an “active” state characterized by a nonzero scalar order parameter to a nondegenerate “absorbing” state provided that (i) the interactions are short range and translation invariant, (ii) the probability at which the transition takes place is strictly positive, and (iii) there are no multicritical points belong to the universality class of directed percolation (DP). In the Domany–Kinzel (DK) cellular automaton [8, 9], which is the simplest PCA of this type, Martins et. al. [10] discovered a new “phase.” These authors considered two initial random configurations differing only at one site, and studied their evolution with identical realizations of the stochastic noise. The site at which the initial

configurations differ may be regarded as a “damage” and the question they addressed was whether this damage will eventually heal or spread. The new phase, they called “chaotic”, is characterized by a nonzero density of damaged sites. In a recent paper, Grassberger [11] (see also Ref. [12]) showed that this new phase transition also belong to the universality class of directed percolation.

More recently, Hinrichsen [13] stressed that in presence of multiple absorbing states with symmetric weight, the universality class changes from DP to a parity conservation (PC) class, with quite different exponents.

The damage-spreading phase transition can be related to usual chaotic properties (i.e. positivity of maximal Lyapunov exponent) of dynamical systems [14]. The boundary of the damaged phase depend in general on the algorithm used for the implementation of the evolution rule. Recently, however, Hinrichsen [15] formulated an objective, algorithm-independent definition of damage spreading transitions. In particular, the algorithm with maximal correlation between the random numbers used in the updating (i.e. using only one random number) allows the computation of the minimum boundary of damage spreading transitions: inside this region a damage will spread whichever al-

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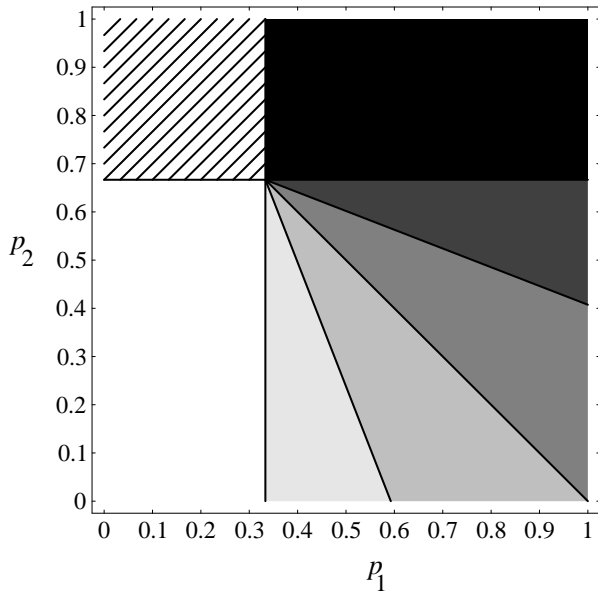


Figure 1: Mean-field phase diagram for the density c of active sites for $p_3 = 1$, as described by Equation (1). The gray levels indicates the asymptotic value of c ; contours for $c = 0, 1/4, 1/2, 3/4$ and 1 . The dashed region marks coexistence of phases. The first order phase boundary depends on the initial condition.

gorithm is used.

In this paper, we study the phase diagram and the critical behavior of a one-dimensional radius-1 totalistic PCA having two absorbing states, and which exhibits a multicritical point. In the DK model, which is a two-input PCA characterized by the transition probabilities $P(1|00) = 1$, $P(1|01) = P(1|10) = p_1$ and $P(1|11) = p_2$, the system has two absorbing states for $p_2 = 1$, and exhibit a first-order phase transition at $p_1 = 1/2$. We may therefore, expect that our system will have a nontrivial phase diagram. Moreover, the discretization of differential equations lead naturally to radius-1 rules (or three-inputs rules) and we feel that it would be useful to have a better knowledge of the phenomenology of simple models formulated in terms of radius-1 PCA.

2 The model

The evolution rule of our model is defined as follows. Let $s(i, t)$ denotes the state of the i -th cell at time t , and $\theta(i, t) = s(i-1, t) + s(i, t) + s(i+1, t)$ the sum of

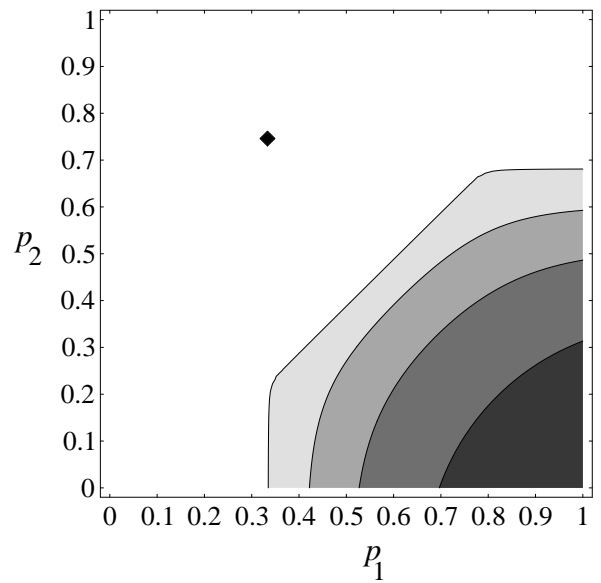


Figure 2: Mean-field damage-spreading phase diagram for $p_3 = 1$. The phase diagram has been obtained numerically iterating Equation (2). Levels of damage are $h = 0, 1/8, 1/4, 3/8$ and $1/3$. The isolated dot marks the multicritical point.

the cells in the neighborhood, then

$$s(i, t+1) = \begin{cases} 0, & \text{if } \theta(i, t) = 0, \\ X_1, & \text{if } \theta(i, t) = 1, \\ X_2, & \text{if } \theta(i, t) = 2, \\ X_3, & \text{if } \theta(i, t) = 3, \end{cases}$$

where X_j ($j = 1, 2, 3$) is a Bernoulli random variable equal to 1 with probability p_j , and to 0 with probability $1 - p_j$. That is, the transition probabilities of this model are $P(0|000) = 1$, $P(1|001) = P(1|010) = P(1|100) = p_1$, $P(1|011) = P(1|101) = P(1|110) = p_2$, and $P(1|111) = p_3$. Cell i is said to be “empty” at time t if $s(i, t) = 0$, and “occupied” if $s(i, t) = 1$. The state in which all cells are empty is a fixed point (absorbing state) of the dynamics. If $p_3 = 1$, the state in which all cells are occupied is also a fixed point. In this work we study mainly the case $p_3 = 1$, delaying the study of the case $p_3 < 1$ to section 5.

If there exists a stable “active” state such that the asymptotic density c of occupied cells is neither zero nor one, then the model will exhibit various bifurcations between these different states as the parameters p_1 and p_2 vary. This totalistic PCA can be viewed as a simple model of opinion formation. It assumes that our own opinion and the opinion of our nearest neighbors have equal weights. The role of social pressure is twofold. If there is homogeneity of opinions, then one does not change his mind (absorbing states),

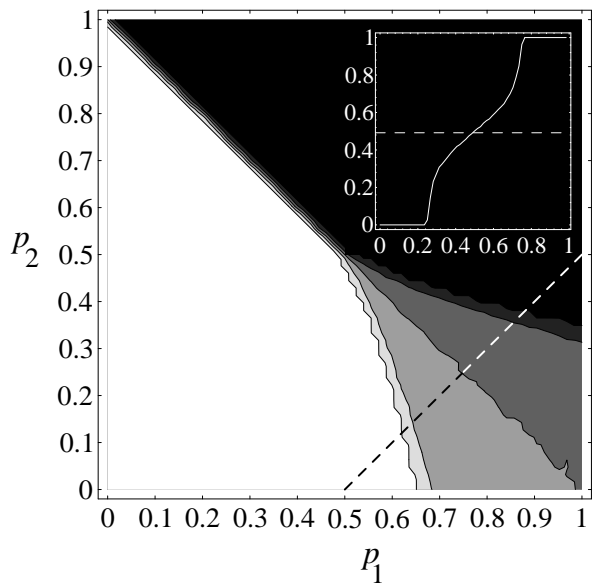


Figure 3: Phase diagram for the density of active sites c obtained by direct simulations. One run was performed. Lattice size: 10000, number of time steps: 10000, resolution: 64 different values for both p_1 and p_2 ($p_3 = 1$), color codes as in Figure 1. The initial density is $c_0 = 0.5$. The inset represents the density profile along the dashed line. Two critical phase transitions are evident.

otherwise one can agree or disagree with the majority with a certain probability.

Let us assume that one starts with a random configuration half filled with 0's and 1's. On the line $p_1 + p_2 = 1$, if p_1 is small, to first order in p_1 , clusters of ones perform symmetric random walks, and have equal probabilities to shrink or to grow. Slightly off this line, the random walks are no more symmetric, and according to whether p_1 is greater or less than $1 - p_2$, all the cells will eventually be either occupied or empty. $p_1 + p_2 = 1$ is, therefore, a first-order transition line. This line, however, cannot extend to $p_1 = 1$ since, for $p_2 = 0$, our model is similar to the diluted XOR rule (Rule 90) of the DK model. We thus expect that our model will exhibit, for a certain critical value of p_1 , a second-order phase transition coinciding with the damage-spreading phase transition. For $p_1 = 1$ and $p_2 = 0$, frustration is large, and our model is just the modulo-2 rule (Rule 150) as in the DK model.

For $p_3 = 1$, the model is symmetric under the exchange $p_1 \leftrightarrow 1 - p_2$, and $0 \leftrightarrow 1$. Thus, along the line $p_2 = 1 - p_1$ the two absorbing states 0 and 1 have equal weight, and, according to Hinrichsen [13], this transition should belong to the PC universality class.

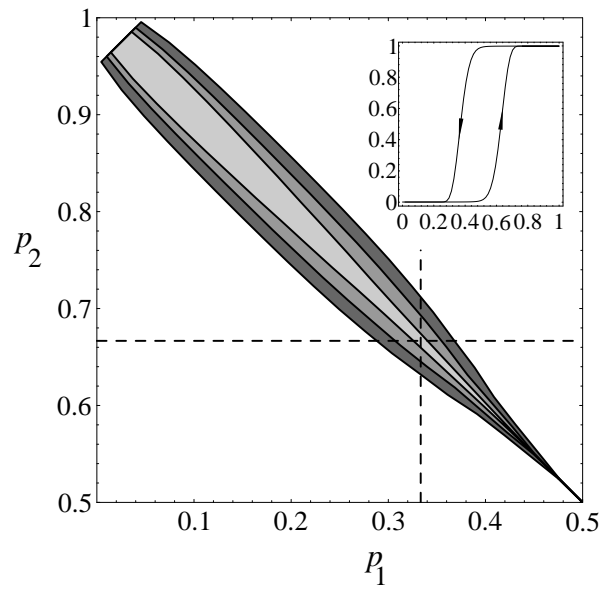


Figure 4: Profile of the hysteresis region for several values of the noise ε and relaxation time T . Data obtained from the local structure approximation of order $l = 6$; the regions corresponds to the intersections of the hysteresis cycle with $c = 0.5$ (horizontal dashed line in the inset). The hysteresis boundary lines smoothly join at $p_1 = 0, p_2 = 1$ (not represented). Darker to lighter areas correspond to $T = 500, \varepsilon = 0.0001$; $T = 1000, \varepsilon = 0.0001$; $T = 500, \varepsilon = 0.001$. The dashed lines represent the mean-field hysteresis region. The inset represents the cycle along a line parallel to the diagonal $p_1 = p_2$.

3 Mean-field approximation

In order to have a qualitative idea of its behavior, we first study our model within the mean-field approximation. If $c(t)$ denotes the density of occupied cells at time t , we have

$$c(t+1) = 3p_1 c(t) (1 - c(t))^2 + 3p_2 c^2(t) (1 - c(t)) + c^3(t). \quad (1)$$

This one-dimensional map has three fixed points:

$$0, \quad 1, \quad \text{and} \quad c^* = \frac{3p_1 - 1}{1 + 3p_1 - 3p_2}.$$

0 is stable if $p_1 < \frac{1}{3}$, 1 is stable if $p_2 > \frac{2}{3}$, and c^* is stable if $p_1 > \frac{1}{3}$ and $p_2 < \frac{2}{3}$. In the (p_1, p_2) -parameter space, the bifurcations along the lines

$$p_1 = \frac{1}{3}, \quad 0 \leq p_2 \leq \frac{2}{3},$$

$$\frac{1}{3} \leq p_1 \leq 1, \quad p_2 = \frac{2}{3},$$

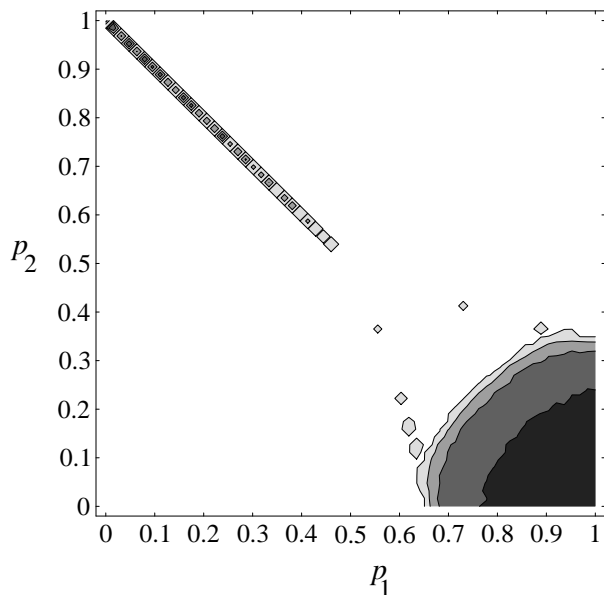


Figure 5: Phase diagram for the damage spreading from direct numerical simulations. Numerical values as in Figure 3, color codes as in Figure 2. Traces of the second order phase transitions are present; they join to the first order ($c_0 = 0.5$) phase boundary.

may be viewed as second-order transition lines; the first one between the active and empty states, and the second one between the active and fully occupied states.

In the domain defined by

$$0 \leq p_1 < \frac{1}{3} \quad \text{and} \quad \frac{2}{3} < p_2 \leq 1$$

fixed points 0 and 1 are both stable. Their basins of attraction are, respectively, the semi-open intervals $[0, c^*[$ and $]c^*, 1]$. Therefore, if we start from a uniformly distributed random value of c , as time t goes to infinity, $c(t)$ tends to 0 with probability c^* , and to 1 with probability $1 - c^*$. Since, for $p_1 + p_2 = 1$, $c^* = \frac{1}{2}$, the line defined by

$$p_1 + p_2 = 1, \quad \text{with} \quad 0 \leq p_1 < \frac{1}{3} \quad \text{and} \quad \frac{2}{3} < p_2 \leq 1$$

is similar to a first-order transition line, although we cannot define a free energy for our dynamical system.

In general, first order phase transitions are associated to the presence of an hysteresis cycle, due to the coexistence of two phases, both in equilibrium. The problem of a proper definition of such a cycle is rather subtle: since this model presents absorbing states, it is out of equilibrium and the ergodicity is always broken, even for finite-size systems. Thus, once the system settles into one of the two absorbing states, it

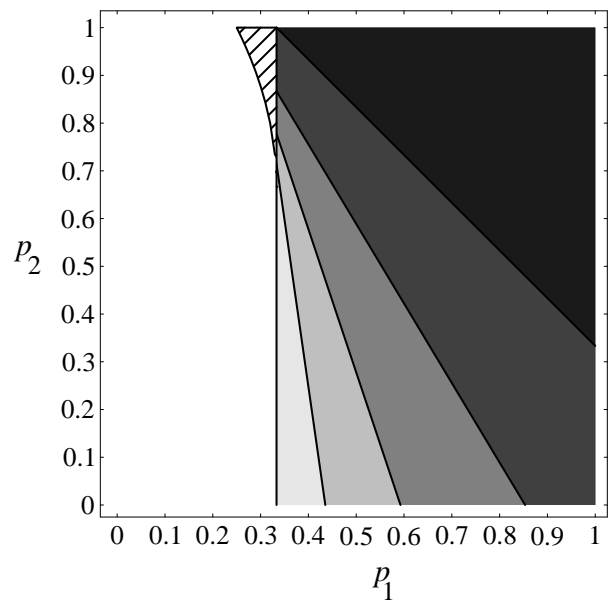


Figure 6: Mean-field phase diagram for $p_3 = 0$. Color codes as in Figure 1.

never exits, even if it is not stable. One can circumvent this problem adding a little of noise, i.e. setting $P(0|000) = 1 - P(1|111) = \varepsilon$. This assumption, however, brings the model into the class of equilibrium models, thus forbidding the presence of a true phase transition in one dimension. However, if ε is small, the system can get trapped for a long time T in a metastable region near an absorbing state. The time T is also a function of the system size L . Thus, we have to perform carefully the limits $L \rightarrow \infty$, $T \rightarrow \infty$ and $\varepsilon \rightarrow 0$. In practice, we have observed that the intensity of noise ε strongly affects the amplitude of the hysteresis region, while, for a system size L sufficiently large, there exists a large interval of possible time length T . The results from numerical simulations are reported in section 4.

In the mean-field approximation the hysteresis region ranges from $0 < p_1 < 1/3$, $p_2 = 2/3$ to $p_1 = 1/3$, $2/3 < p_2 < 1$. The mean-field phase diagram for the density is shown in Figure 1.

One can write down the mean-field equation for the damage spreading taking into consideration all possible local configurations of the two lattices. Let us denote by $\eta(t)$ the density of damaged sites at time t . The evolution equation for η depends on correlations among sites, i.e. on the order of the mean field for the density. Using the simplest factorization for the density, $\eta(t)$ depends on $c(t)$. The evolution equation for the minimum damage η , i.e. the damage when the evolution of the two replicas is computed using only one

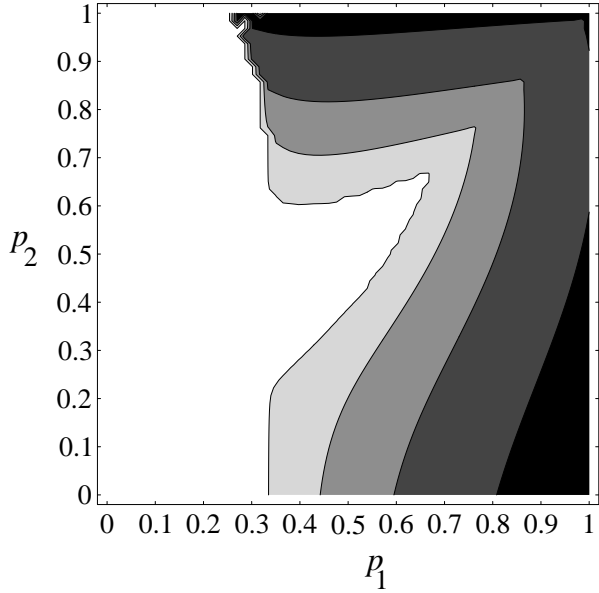


Figure 7: Mean-field damage-spreading phase diagram, for $p_3 = 0$. The diagram has been obtained numerically iterating Equation (2).

random number, is given by

$$\eta(t+1) = \sum_{\substack{s_1 s_2 s_3 \\ h_1 h_2 h_3}} \pi(c(t), s_1 s_2 s_3) \pi(\eta(t), h_1 h_2 h_3) \cdot |P(1|s_1 s_2 s_3) - P(0|s_1 s_2 s_3 \oplus h_1 h_2 h_3)| \quad (2)$$

where

$$\pi(\alpha, x_1 x_2 x_3) = \alpha^{x_1 + x_2 + x_3} (1 - \alpha)^{3 - x_1 + x_2 + x_3}$$

and the symbol \oplus represents the bitwise sum modulus two (XOR) of two Boolean configurations. The value for $c(t)$ is given by mapping (1).

We have numerically iterated Equation (2). The resulting mean-field phase diagram for the damage is shown in Figure 2.

4 Numerical simulations

We used the fragment method [16] to determine the phase diagram. Figure 3 represent a plot for the asymptotic density c of occupied cells when the initial fraction of the occupied cells is equal to 0.5. The scenario is qualitatively the same as predicted by the mean-field analysis. In the vicinity of the point $(p_1, p_2) = (0, 1)$ we observe a discontinuous transition from $c = 0$ to $c = 1$, while close to $(p_1, p_2) = (1, 0)$ the transition is continuous. The two second-order phase-transition lines from the non fully occupied states to either the fully occupied state or the empty state are symmetric and the critical behavior of the respective

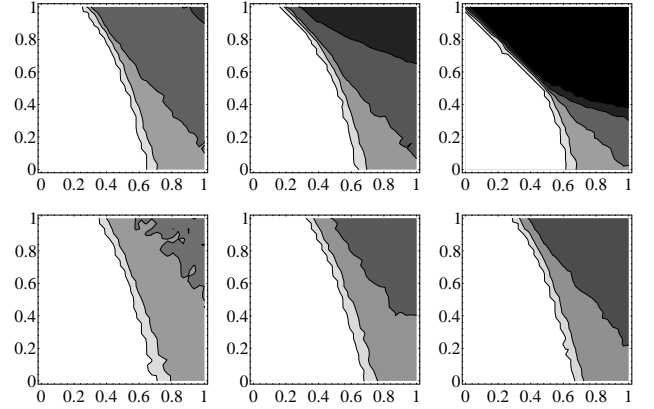


Figure 8: Cut of the density phase space along the p_1 - p_2 plane for different values of p_3 . From left to right and for bottom to top: $p_3 = 0.0, 0.2, 0.4, 0.6, 0.8$ and 1.0 . Color codes as in Figure 1

order parameters, $1 - c$ and c , are the same. We have checked that the critical exponent β for these transitions far from the crossing point are numerically the same of DP.

These two transition lines meet on the diagonal $p_1 + p_2 = 1$ and become a first-order transition line. Crossing the phase boundaries on a line parallel to the diagonal $p_1 = p_2$, the density c exhibits two critical transitions, as shown in the inset of Figure 3. Approaching the crossing point, the critical regions of the two transitions vanishes, the transition itself become sharper and the corrections to scaling increase.

At the crossing point, around $(0.5, 0.5)$, the two attractors have symmetrical weight. If we relabel couples of 1's with the symbol 1, couples of 0's with the symbol 0 and couples 01 or 10 with the symbol A , we fulfill the condition stated by Hinrichsen [13] to have symmetric absorbing states, whose cluster are always separated by an active (A) layer. We have performed preliminary measurements of the exponent β for the density of A couples along the line $p_1 + p_2 = 1$, and obtained $\beta \simeq 0.65(5)$, which is consistent with the values for the PC process reported in Refs. [17, 18]. The transition point is at $p_1 = 0.460(2)$, which defines the position of the tricritical point.

In order to study the presence of an hysteresis region, we performed several scanning of the first-order transition using the local structure approximation [19]. We cut the phase diagram with a line parallel to the diagonal $p_1 = p_2$, increasing the value of p_1 and p_2 after a given relaxation time up to $p_2 = 1$; then reverting the scanning up to $p_1 = 0$. The results are reported in Figure 4. The size of the hysteresis region grows with the level of the noise ε . Preliminary simulations show

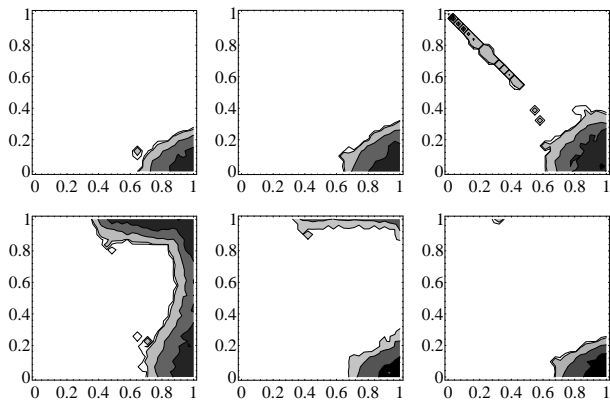


Figure 9: Cut of the damage-spreading phase space along the p_1 - p_2 plane for several values of p_3 . From left to right and for bottom to top: $p_3 = 0.0, 0.2, 0.4, 0.6, 0.8$ and 1.0 . Color codes as in Figure 2.

that the relaxation time T and the noise level ε scales as $\varepsilon T^{1.434} = \text{const.}$

The phase diagram for the damage spreading is shown in Figure 5. Outside the true damage-spreading region, there appear small damaged domains on the other phase boundaries. This is due either to the divergence of the relaxation time (second-order transitions) or to the fact that a small difference in the initial configuration can drive the system to a different absorbing state (first-order transitions).

The “chaotic” domain near the point $(p_1, p_2) = (1, 0)$ is stable regardless of the initial density. Note that on the line $p_2 = 0$ its boundary coincides with the transition line from the active state to the empty state. Using an argument similar to the one in Ref. [12] would prove that the derivatives of the two boundary curves coincide, that is, the chaotic phase exhibits a *reentrant* behavior. Here again, the symmetry of the phase diagram implies a similar behavior on the line $p_1 = 1$.

5 Extended phase diagram

It is interesting to study our model when the transition probability $P(1|111) = p_3 < 1$. In this case, there is only one absorbing state.

5.1 Mean-field approximation

For some values of p_3 , the mean-field phase diagram still exhibits a first-order phase transition. Let us first examine in detail the particular case $p_3 = 0$. As for $p_3 = 1$, the solution $c = 0$ loses its stability at $p = 1/3$. The other solutions for the density c are the roots

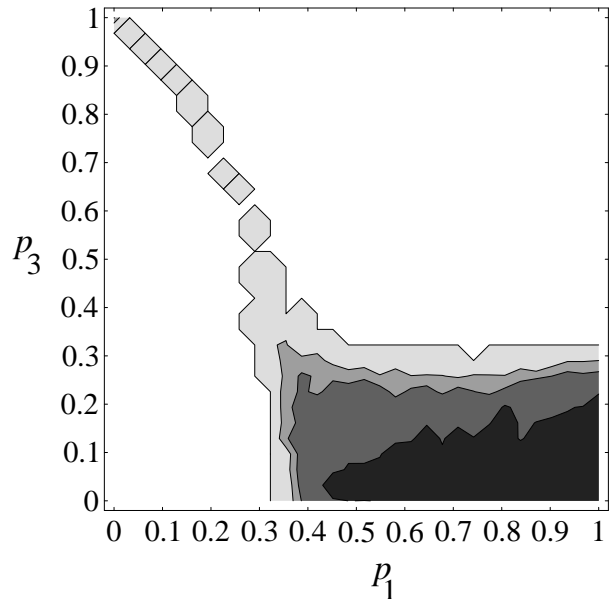


Figure 10: Cut of the density phase space along the p_1 - p_3 plane for $p_2 = 1$. Color codes as in Figure 1.

of the quadratic equation

$$c^2(p_1 - p_2) + c(p_2 - 2p_1) + p_1 - \frac{1}{3} = 0.$$

These two roots are real if $q^2 + 4(p_1 - p_2)/3 \geq 0$. This is always the case if $p_1 > 1/3$, while for $p_1 < 1/3$, p_2 must lie outside the interval $[\frac{2}{3}(1 - \sqrt{1 - 3p_1}), \frac{2}{3}(1 + \sqrt{1 - 3p_1})]$. Since the density c has to be such that $0 \leq c \leq 1$, only the condition $1 \geq p_2 > \frac{2}{3}(1 + \sqrt{1 - 3p_1})$ is meaningful. These solutions are stable. Therefore, there is a domain in which two stable solutions coexist. The boundaries of this domain are the three straight lines $p_1 = \frac{1}{3}$, $p_2 = 1$ and $p_2 = \frac{2}{3}(1 + \sqrt{1 - 3p_1})$, where for this last line p_2 belongs to the interval $[\frac{1}{4}, \frac{1}{3}]$. When, inside the domain, we approach the point $(p_1, p_2) = (\frac{1}{3}, \frac{2}{3})$, the density c tends continuously to zero, which shows that, at this point, the transition is second-order. The corresponding phase diagram is represented in Figure 6.

For the chaotic phase, the boundary of the domain of stability is shown in Figure 7. Close to the point $(p_1, p_2) = (1, 0)$ the boundary is almost the same as for $p_3 = 1$ but there exists another domain close to the point $(p_1, p_2) = (1, 1)$ which did not exist for $p_3 = 1$. In the vicinity of $(1, 1)$, very small clusters, which tend to grow, are then fragmented because $p_3 = 0$. We have also studied the case $0 < p_3 < 1$.

5.2 Numerical simulations

Our numerical simulations give results in good qualitative agreement with the mean-field approximation. Figure 8 shows different phase diagrams in the (p_1, p_2) -plane corresponding to different values of p_3 . As expected, the boundary of the active phase ($c \neq 0$) for $p_2 = 0$ does not vary much with p_3 since, in this case, the probability to find a cluster of three or more one is vanishing. No first order phase transition for the density is apparent by these simulations.

Figure 9 shows the domain of stability of the chaotic phase in the (p_1, p_2) -plane for different values of p_3 . The domain of stability of the chaotic phase in the (p_1, p_3) -plane for $p_2 = 1$ is represented in Figure 10. One can see that there are damage-spreading domains corresponding to the density phase boundaries.

6 Conclusion

We have studied a radius-1 totalistic PCA whose transition probabilities are given by $P(1|000) = 0$, $P(1|001) = P(1|010) = P(1|100) = p_1$, $P(1|011) = P(1|101) = P(1|110) = p_2$, and $P(1|111) = p_3$, within the framework of the mean-field approximation and using numerical simulations.

For $p_3 = 1$, the system has two absorbing states and four different phases: an empty state in which no cell is occupied, a fully occupied state, an active phase in which only a fraction of the cells are occupied, and a chaotic phase. This system exhibits:

1. A first-order phase transition from the fully occupied state to the empty state.
2. Two second-order phase transitions from the active state to the fully occupied state and from the active state to the empty state.
3. A second-order phase transition from the active state to the chaotic state.

For $p_3 < 1$, our numerical simulations show that there is no first-order phase transition while the mean-field approximation still predicts the existence of such a transition.

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